SPANNING TREE CONGESTION, STRETCH AND OTHER PARAMETERS OF THE GRAPH

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Abstract

In this article we introduce a new characteristic of connected simple finite graphs which we term *support number*. We present relations with the previously defined notions of *congestion* and *stretch*. We also give an approximation algorithm for support number, and thus a polynomially computable lower bound for stretch.

Congestion and Stretch

The following definitions leading to the notion of *spanning tree congestion* were developed in [**Ost04**]. Let G be a connected simple finite graph. Let T be a spanning tree in G. For an edge e of T let A_e and B_e be the vertex sets of the components of $T - e$.

1. The number $e_G(A_e, B_e)$ of edges in G that connect a vertex in A_e to a vertex in B_e is called *the congestion of G in T at edge e*.
2. The *edge congestion* of G in T is
$$\mathrm{ec}(G:T) = \max_{e \in E(T)} e_G(A_e, B_e).$$
3. The *spanning tree congestion* of G is
$$s(G) = \min_T \mathrm{ec}(G:T).$$

A more classical concept is that of *stretch* (see [**PU89**]).

1. Let H be a connected spanning subgraph of G. The *stretch* of H in G is defined by
$$\mathtt{Stretch}_G(H) = \max_{u,v \in V(G)} \frac{d_H(u,v)}{d_G(u,v)}$$
2. The *stretch* of G is defined by minimizing with respect to its spanning trees
$$\mathtt{Stretch}(G) = \min_T \mathtt{Stretch}_G(T)$$

We defined two min-max characteristics of G via the family of its spanning trees

$$\boxed{s(G) = \min_T \max_{e \in E(T)} e_G(A_e, B_e)}$$

and

$$\boxed{\mathtt{Stretch}(G) = \min_T \max_{u,v \in V(G)} \frac{d_T(u,v)}{d_G(u,v)}}$$

Congestion-Stretch duality in plane graphs

The dual graph G^* of a plane graph G is the multigraph whose vertices correspond to the faces of G, including the exterior face. Two vertices of G^* are joined by an edge if and only if their corresponding faces in G have a common edge in their boundaries.

1. If T is a spanning tree of a plane graph G, then the dual spanning tree $T^\sharp$ is defined as the spanning subgraph of G^* such that $e^* \in E(T^\sharp)$ if and only if $e \notin E(T)$.

LEMMA: Let G be a connected planar graph with dual graph G^*. Then

$$\boxed{\mathrm{ec}(G) = \mathtt{Stretch}(G^*) + 1}$$

Grids

Equivalence of congestion-stretch characteristics for plane graphs yields dual approaches for finding their exact values for particular plane grids.

Rectangular Grid

Let P_m be the path with m nodes, and consider the grid $G = P_m \times P_n$ $(2 \le m \le n)$.
The exact value for congestion was computed in [**Hru08**] and [**Ost10**], while working on the stretch side it was verified in [**LL17**].

THEOREM (see [**LL17**]): For the rectangular grids $P_m \times P_n$ with $2 \le m \le n$, we have

$$\mathtt{Stretch}(P_m \times P_n) = 2\left\lfloor \frac{m}{2} \right\rfloor + 1.$$

Triangular Grid

Let T_n be the graph obtained by dividing each side of a triangle into $(n-1)$ pieces and joining the corresponding subdivision points of the different sides of triangle. In these graphs all intersection points of line segments are regarded as vertices and there are no other vertices.

THEOREM (see [**LL17**]): For triangular grids T_n it holds that

$$\mathtt{Stretch}(T_n) = \left\lceil \frac{2n}{3} \right\rceil + 1.$$

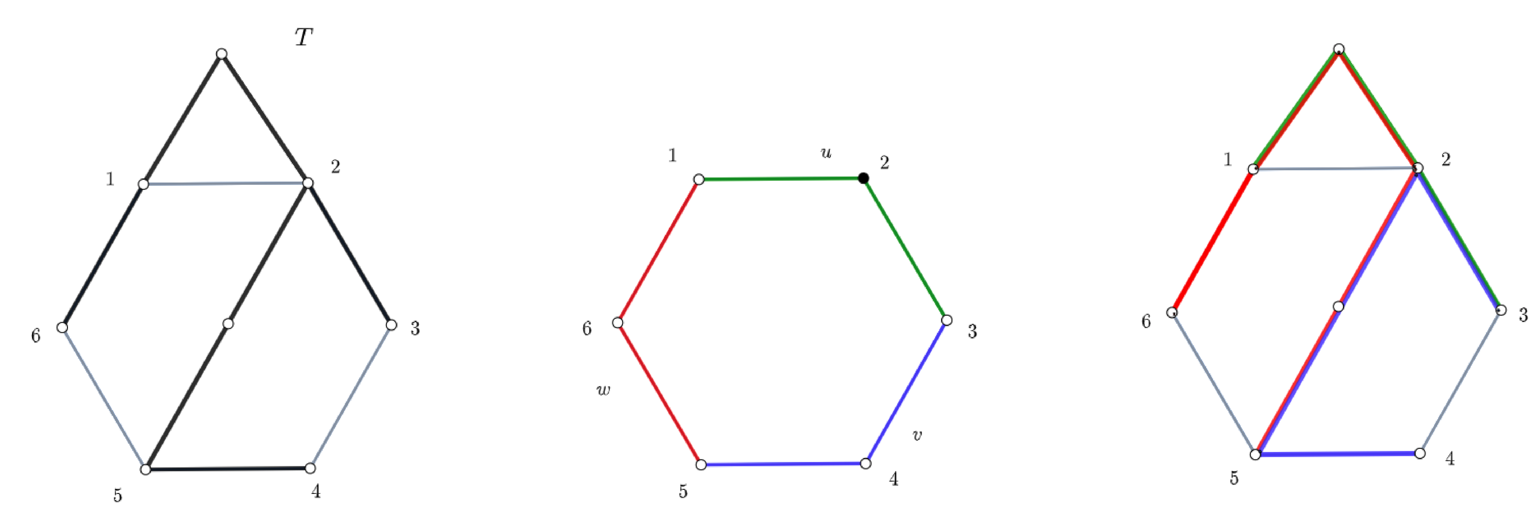
k-supported Cycles

In this work we introduce two new characteristics of finite, connected, simple graphs. These notions are closer related to that of stretch than to congestion. The first is that of *support number*.

1. If a cycle C in G can be partitioned into three intervals I_1, I_2, and I_3, in such a way that for every triple (u_1, u_2, u_3) of vertices satisfying $u_i \in I_i$, $i = 1, 2, 3$, we have
$$\max_{i,j \in \{1,2,3\}} d_G(u_i, u_j) \ge k,$$
we say that C is a *k-supported cycle*.

THEOREM: If a graph G contains a k-supported cycle, then

$$\mathtt{Stretch}(G) \ge k$$



a) b) c)

The key step in the proof of the THEOREM is the following

PROPOSITION (see [**RR98**]): Let $f : \mathbb{S}^1 \to T$ be a continuous map, and let $\{I_1, I_2, I_3\}$ be an arbitrary partition of $\mathbb{S}^1$ into three intervals with mutually disjoint interiors. Then there exists $c \in T$ such that $f^{-1}(c)$ has a representative in each of these intervals.

COROLLARY: If a graph G contains an isometrically embedded C of length n, then

$$\mathtt{Stretch}(G) \ge \left\lfloor \frac{1}{3} n \right\rfloor$$

We remark that finding whether isometrically embedded cycles of a given length exist can be done in polynomial time (see [**Lok09**]).

Cycle Width

The second characteristic introduced in this work is that of *cycle width*. For a fixed vertex r of G consider a breadth-first search tree rooted at r. For $x \in V(G)$ denote the *level* of x by $\ell(x) = d_G(r, x)$. For each $n \in \mathbb{N}$ consider the subset $R(n)$ of edges in G consisting of edges xy for which

$$\max\{\ell(x), \ell(y)\} > n.$$

1. The *cycle width* of G is $W(G) = \max_{r \in V(G)} W(r)$ where
$$W(r) = \max_n \max_{\ell(x)=\ell(y)=n} d_G(x,y),$$
and x and y are in the same component of $(V(G), R(n))$.

THEOREM:

A. Each graph G contains a polynomially computable $W(G)/3$-supported cycle.

B. Each graph G containing a k-supported cycle satisfies $W(G) \ge k - 3$.

Remarks

It is known that the problem of finding the minimum stretch spanning trees in a graph is NP-hard (see [**CC95**], [**FK01**]). We conjecture that finding the support number and k-supported cycles with largest k is also NP-hard. Since the cycle width we introduced is computable in polynomial time, Theorems A. and B. provide useful estimates.

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